Polar-Form Proof of the Euler Parameterization for a General 2×2 Unitary
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1 Introduction

We show that any $U \in U(2)$ can be written (up to an overall phase) as

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

with real angles θ, ϕ, λ.

2 Selecting First Column

Let the first column of U be a normalized complex vector

$$\mathbf{u} = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}, \quad |u_{11}|^2 + |u_{21}|^2 = 1.$$

Write each entry in polar form:

$$u_{11} = r_1 e^{ia}, \qquad u_{21} = r_2 e^{ib}, \qquad r_1^2 + r_2^2 = 1.$$

Introduce a single angle $\theta \in [0, \pi]$ by

$$r_1 = \cos\frac{\theta}{2}, \qquad r_2 = \sin\frac{\theta}{2},$$

leading to

$$\mathbf{u} = \begin{pmatrix} e^{ia}\cos\frac{\theta}{2} \\ e^{ib}\sin\frac{\theta}{2} \end{pmatrix}.$$

and we factor out an immaterial global phase e^{ia} and relabel $\phi = b - a$, $\phi \in [0, 2\pi]$ with the result

$$\mathbf{u} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

3 Orthogonalization

A unit vector orthogonal to $\mathbf{u}$ is (do not forget complex conjugation for one of the vectors)

$$\mathbf{v} = \begin{pmatrix} -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{ib}\cos\frac{\theta}{2} \end{pmatrix},$$

since $\mathbf{u}^\dagger\mathbf{v} = 0$ and $|\mathbf{v}| = 1$, we obtain one more constraint between the phases, $\lambda = b - \phi$, and we obtain

$$\mathbf{v} = \begin{pmatrix} e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}, \qquad c \in \mathbb{R}$$

and combining the two vectors to form the 2×2 unitary leads to the final result:

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{\,i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

consistent with the 2×2 unitary parameterization in the main text and in the simulator.

4 Completeness

Every first column of a unitary is some normalized complex vector of the form

$$\left(r_1,\, r_2\, e^{i\phi}\right)^\top = \left(\cos\frac{\theta}{2},\, -e^{i\phi}\sin\frac{\theta}{2}\right)^\top$$

covers *all* such vectors because (r_1, r_2) run over the unit circle via θ, and ϕ provides an independent phase. Given this arbitrary first column, the orthonormal second column is unique up to an irrelevant global phase. Therefore, the construction reaches *every* 2×2 unitary.

5 Parameter Count

A general 2×2 complex matrix has four complex elements, corresponding to eight real parameters. The unitarity condition $U^\dagger U = I$ imposes four independent real constraints (two from normalization of columns and two from orthogonality), reducing the count to four free parameters. The remaining global complex phase, $\det U = e^{i\epsilon}\,\epsilon real$, does not affect observable quantities and can be factored out as a global phase. This leaves three independent real parameters, θ, ϕ, and λ, consistent with the explicit Euler-angle representation for an arbitrary unitary matrix given in the main text.

6 Conclusion and Outlook

In closing we would like to emphasize that you may find this parameterization in many different context. For example in the quantum mechanical description of a beamsplitter [1]; and in the description of binary logical gates in quantum computing [2].

References

[1] Christopher C Gerry and Peter L Knight. *Introductory quantum optics.* Cambridge University Press (Virtual Publishing), Cambridge, England, 2 edition, January 2024.

[2] Michael A Nielsen and Isaac L Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition, Cambridge University Press, pp. 702.* Cambridge University Press, 1 edition, January 2011.