Vectors

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1 Motivation

In physics or more generally in science many quantities are not just numbers, but they have both a magnitude and a direction. For example, saying "the library is 0.5 km away" gives a scalar distance (magnitude), but it does not tell us which direction to the library. To fully specify the location of the library, we need a vector, which provides both magnitude (scalar direction) and direction. Some other examples of vectors are force, velocity, acceleration, magnetic, and electrical fields.

2 Definition of Vectors

A vector is specified by one for each dimension, in two dimensions a vector has two elements, in three dimensions a vector has three elements, and a N-dimensional vector has N entries. To be specific in two dimensions we write a vector as an ordered pair:

$$\overrightarrow{T} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

where t_1 represents displacement along the horizontal (x-axis, $\overrightarrow{e}_1$); t_2 represents displacement along the vertical (y-axis, $\overrightarrow{e}_2$). These two numbers define any direction in the plane, much like points on a compass. Moreover, with this interpretation we can provide an alternative representation of vectors:

$$\overrightarrow{T} = t_1 \cdot \overrightarrow{e}_1 + t_2 \cdot \overrightarrow{e}_2$$

that emphasizes the components as well as the directions. In three dimensions, a vector has three components:

$$\overrightarrow{T} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

$$\overrightarrow{T} = t_1 \cdot \overrightarrow{e}_1 + t_2 \cdot \overrightarrow{e}_2 + t_3 \cdot \overrightarrow{e}_3$$

where now t_3 accounts for motion along a third z-direction, giving us full spatial positioning. And for an N-dimensional vector

$$T = \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}$$
$$\overrightarrow{T} = t_1 \cdot \overrightarrow{e}_1 + \cdots + t_N \cdot \overrightarrow{e}_N$$

3 Basis Vectors

If a set of directional vectors, $\{\overrightarrow{e}_1, \cdots, \overrightarrow{e}_N\}$, allows to identify any point in the corresponding space, this set is called a basis. For example, the set,

$$\left\{\overrightarrow{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \overrightarrow{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$
$$\overrightarrow{T} = t_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

allows to reach all points in the two-dimensional plane, and the two vectors form a basis. On the other hand,

$$\left\{\overrightarrow{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \overrightarrow{e}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right\}$$
$$\overrightarrow{T} = t_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

do not form a basis, since the two vectors point in the same direction. We notice in the first example, that the two directional vectors are orthogonal. However, this feature is not a prerequisite for vectors to form a basis:

$$\left\{\overrightarrow{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \overrightarrow{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$
$$\overrightarrow{T} = t_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

However, orthogonal basis vectors simplify often the math and usually it is not a restriction generality to assume a n orthogonal set of vectors. Note, that even if we start with a non-orthogonal set, we can always transform this set (uniquely) into an orthogonal set.

4 Vector Algebra

Vectors can be manipulated in several ways, they can be added and subtracted and multiplied by scalars to give another vector

$$\vec{T_1} = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}; \vec{T_2} = \begin{pmatrix} T_{12} \\ T_{22} \end{pmatrix}$$
$$\overrightarrow{T}' = s \cdot \overrightarrow{T}_1 + t \cdot \overrightarrow{T}_2$$
$$\overrightarrow{T}' = \begin{pmatrix} sT_{11} + tT_{12} \\ sT_{21} + tT_{22} \end{pmatrix}$$

and here is a specific example:

$$\vec{T_1} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}; \vec{T_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$s = 2, t = -2$$
$$\overrightarrow{T}' = \begin{pmatrix} 2 * 2 - 2 * 1 \\ 2 * (-4) - 2 * 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

5 Angular Representation of 2D Vectors

If we think about the vectors for a moment, we realize that it must be possible to write a vector in terms a suitably chosen angle, θ, by convention chosen as counterclockwise from the x-axis, and its magnitude, $|\overrightarrow{T}|$. Using the definitions of sin/cos (for a review, see "Trigonometry Primer")

$$T_1 = T \cdot cos(\theta)$$
$$T_2 = T \cdot sin(\theta)$$

where $T = |\overrightarrow{T}|$ is the magnitude of the vector. For example:

$$\theta = 5\pi/4 = 225^o; T = 3$$
$$T_1 = 3 * (-0.707) = -2.121$$
$$T_2 = 3 * (-0.707) = -2.121$$

Let's complete the cycle and re-compute angle and magnitude from the polar representation. Here we have to be careful, since the angle may not be uniquely defined. In our example, we could compute

$$tan\,(\theta) = \frac{T_2}{T_1} = 1$$

$$\theta = tan^{-1}\left(\frac{T_2}{T_1}\right) = \pi/4 = 45^o$$

Thus, computing the component representation from the polar representation is straightforward and unique, the same is not true for computing the polar representation from the component representation and we have to be careful to pick the correct angle. The tan-function is π-periodic, and adding 180^o to the computed angle leads to the original angle.

6 Dot-Product

The dot-product combines to vectors, $\overrightarrow{a}$ and $\overrightarrow{b}$, to form a scalar. More precisely, the dot-product is defined as the projection of the second vector onto the first vector. This must always be possible, consider the 2D case: the two vectors define a unique plane, and within this plane exists a unique projection of one vector onto the other vector.

$$\overrightarrow{a} \cdot \overrightarrow{b}$$
$$= a \cdot b \cdot cos\,(\theta)$$
$$= a_1 b_1 + a_2 b_2$$

where, a and b, are the magnitude of the two vectors,

$$\overrightarrow{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

and θ is the angle enclosed between the two vectors. Here is an example

$$\overrightarrow{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \overrightarrow{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$a = 1; b = \sqrt{2}$$
$$1 \cdot 1 + 0 \cdot 1 = 1 \cdot \sqrt{2} \cdot cos\,(\theta)$$
$$cos\,(\theta) = \frac{1}{\sqrt{2}}$$
$$\theta = \pi/4 = 45^o$$

which is the expected angle. And for the dot-product in N-dimension

$$\overrightarrow{a} \cdot \overrightarrow{b}$$
$$= a \cdot b \cdot cos\,(\theta)$$
$$= \sum_{i=1}^{N} a_i b_i$$

an example where you will find the dot product in physics is when you compute the work done on or by a process.

7 Vector-Product

The vector product, $\overrightarrow{c}=\overrightarrow{a}\times\overrightarrow{b}$ takes two vectors, $\overrightarrow{a}$ and $\overrightarrow{b}$, and creates a new vector, $\overrightarrow{c}$, that is perpendicular to the plane spanned by the two input vectors

$$\overrightarrow{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix};\overrightarrow{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\overrightarrow{c}$$

$$=\overrightarrow{a}\times\overrightarrow{b}$$

$$=\begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$$

$$=a\cdot b\cdot sin\left(\theta\right)$$

and we find that the vector product changes sign if the two input is exchanged

$$\overrightarrow{a}\times\overrightarrow{b}=-\overrightarrow{b}\times\overrightarrow{a}$$

and here is an example

$$\overrightarrow{a}=\begin{pmatrix}1\\0\\0\end{pmatrix};\overrightarrow{b}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\overrightarrow{a}\times\overrightarrow{b}=\begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$|\overrightarrow{a}\times\overrightarrow{b}|=2$$

and the magnitude of the cross-product is equivalent to the parallelogram area spanned by the two input vectors. More generally, the magnitude of the cross product is

$$|\overrightarrow{c}|=|\overrightarrow{a}\times\overrightarrow{b}|$$
$$=a\cdot b\cdot sin\left(\theta\right)$$

In physics you will find the vector when you compute angular momentum, torque and other quantities.

8 Exercises

1. Compute $\overrightarrow{c} = 2 \cdot \overrightarrow{a} - 5 \cdot \overrightarrow{b}$, with $\overrightarrow{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2. Compute the component representation of a 2-dimensional vector with magnitude 4 and an angle of 190^o, measured counterclockwise from the x-axis.

3. Compute the polar representation of the vector, $\overrightarrow{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

4. Compute the polar representation of the vector, $\overrightarrow{a} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

5. Compute the dot product for the two vectors, $\overrightarrow{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, using both definitions of the dot product and show that the results are identical.

6. Compute the vector product, $\overrightarrow{a} \times \overrightarrow{b}$, for the two vectors, $\overrightarrow{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and verify that the magnitude of the resulting vector agrees with the area definition.

7. Compute the area spanned by the parallelogram spanned by the two vectors $\overrightarrow{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.