Statistics

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1 Motivation

In physics, every measurement you make—from timing a pendulum swing to detecting subatomic particles—carries uncertainty, and statistics provides the rigorous framework to quantify and interpret those uncertainties, distinguish signals from random noise, and test theoretical predictions against experimental data; by mastering concepts like probability distributions, confidence intervals, hypothesis testing, and error propagation, you not only learn to report your results with honesty and precision but also gain the tools to uncover subtle patterns, and optimize experimental design. The field of statistics is very large and you find likely complete courses on statistics at your institution. In this primer we will only be concerned with a few fundamental statistical concepts: average, variance, standard deviation, and standard error.

2 Definition of Average

Given a set of N measurements, $x_1, x_2, ..., x_N$, the average (or mean) is the value you compute as:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

From this formula we can infer that the average lies within the value range of the measurements. Moreover, we infer that the average does not have to coincide with measurement value. In this sense, the average captures the center of the data. Also note, that the average converges to a finite value if you do more measurements since we divide by the sample size. For example, if you measure the period of a pendulum five times, $\bar{x}$ is the best single-value estimate of the true period of the pendulum.

3 Variance and Standard Deviation

The variance shows how widely the samples are scattered around the average:

$$Var(x) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

Note, that $Var(x)$ does not have the same units as x, a deficiency that is corrected with the definition of the standard deviation:

$$\sigma = \sqrt{Var(x)}$$

For example, the standard deviation would be the value that you quote in a lab report. Note, this formula supports the intuition that $Var(x)$ and σ are small if the samples scatter little around the average value. Also note, that variance and standard deviation capture the distribution of samples. Therefore, as the number of samples increases, their distribution is expected to converge to a well-defined limiting distribution for $(N \to \infty)$ and variance and standard deviations as measures of this distribution will converge to finite values. For completeness, here is a second equivalent way to compute the variance:

$$\begin{aligned}
Var(x) &= \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i^2 - 2\bar{x}x_i + \bar{x}^2\right) \\
&= \frac{1}{N-1}\left(\sum_{i=1}^{N}x_i^2 - \sum_{i=1}^{N}2\bar{x}x_i + \sum_{i=1}^{N}\bar{x}^2\right) \\
&= \frac{1}{N-1}\left(N\overline{x^2} - 2N\bar{x}^2 + N\bar{x}^2\right) \\
& \frac{N}{N-1}\left(\overline{x^2} - \bar{x}^2\right)
\end{aligned}$$

Let me make another comment on the meaning of the standard deviation of a collection of samples. Could we have chosen 2σ,... instead of σ to characterize the variation of a sample about its average? The answer is yes, and can be clarified for samples that are normally distributed (follow a Bell curve). In this case 1σ means that $\approx 68\%$ of the samples are in the range $\pm 1\sigma$ of the average, $\bar{x}$; case 2σ means that $\approx 95\%$ of the samples are in the range $\pm 2\sigma$ of the average, $\bar{x}$; case 3σ means that $\approx 99.7\%$ of the samples are in the range $\pm 3\sigma$ of the average, $\bar{x}$. This finding supports the intuition that for a larger allowed range we have a higher probability to find a sample value in this range. Note also, that in the limit of allowing a very large range we should find 100% of the samples within that range (as expected).

4 Standard Error

When we repeat an experiment N times and compute $\bar{x}$, the standard error (SE) quantifies how much $\bar{x}$ would fluctuate if we would repeat the experiment over and over again:

$$SE\left(\bar{x}\right) = \frac{\sigma}{\sqrt{N}}$$

this definition supports the intuition that SE should decrease as the number of samples, N, increases.

5 Example

Let's compute the average, variance, standard deviation, and standard error for the following five measurements

$$\{10.1, 10.2, 10.3, 10.4, 10.5\}$$

we obtain for the average:

$$\bar{x} = \frac{1}{5}\left(10.1 + 10.2 + 10.3 + 10.4 + 10.5\right) = 10.3$$

for the variance:

$$Var(x) = \frac{1}{4}\left((10.1 - 10.3)^2 + (10.2 - 10.3)^2 + (10.3 - 10.3)^2 + (10.4 - 10.3)^2 + (10.5 - 10.3)^2\right) \\ = 0.025$$

for the standard deviation:

$$\sigma = \sqrt{0.025} \approx 0.158$$

and for the standard error:

$$SE(\bar{x}) = \frac{\sigma}{5} \approx 0.071$$

and for completeness, here are the details for the computation of the variance using the second approach:

$$\bar{x^2} = \frac{1}{5}\left(10.1^2 + 10.2^2 + 10.3^2 + 10.4^2 + 10.5^2\right) \\ = 106.11$$

and we re-compute the sample variance:

$$\begin{aligned} Var(\bar{x}) &= \frac{5}{4}\left(106.11 - 10.3^2\right) \\ &= 0.025 \end{aligned}$$

the same as before.

6 Exercises

1. Assuming a sample set consists of all integers in the range of 1 to 1000, compute average, variance, standard deviation, and standard error.

2. Assume that you measure 1000 times and you measure -0.5 500 time and +0.5 500 times. What are the average, variance, standard deviations, and standard error.

3. Use a uniform random number generator and draw 100 random numbers in the range [0,1[. Compute standard deviation, and standard error. Repeat the computations with 1000, 10000, 100000 sample sizes. Discuss the size dependence of the standard deviation and the standard error.