Matrices

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1 Motivation

Matrix algebra is a fundamental tool in physics, providing a compact and elegant way to describe and manipulate physical systems. Here are some examples in physics where you will find matrices:

- Classical Mechanics: Rotation matrices are used to describe rotations of coordinate systems and rigid bodies.
- Quantum Mechanics: Operators such as the Hamiltonian are represented as matrices. The eigenvalues of these matrices correspond to measurable energy levels, and eigenvectors describe the states of the quantum system.
- Electromagnetism: Matrices can represent the transformation of fields under rotations and boosts.
- General Relativity: Tensors, which generalize matrices, describe the curvature of space-time and the distribution of matter and energy.

2 Definition of a Matrix

A matrix is a rectangular array of numbers (or, more generally, mathematical objects) arranged in rows and columns. Matrices are often used to represent linear transformations, systems of linear equations, and data sets. For example, a matrix A with m rows and n columns can be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_ij is located in the i-th row and the j-th column.

3 Matrix Operations

Understanding how to operate on matrices is central to matrix algebra. Here are the fundamental operations:

3.1 Matrix Addition and Subtraction

Two matrices can be added or subtracted if they have the same dimensions. This process is performed element-wise:

$$(A + B)_{ij} = a_{ij} + a_{ij}$$

Here is an example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$
$$A + B = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

3.2 Scalar Multiplication

Multiplying a matrix by a scalar (a single number) means multiplying every element of the matrix by that scalar:

$$cA = (cA)_{ij} = c \cdot a_{ij}$$

or with the previously defined matrix A:

$$4 \cdot A = 4 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}$$

3.3 Matrix Multiplication

Two matrices A (of size $m \times n$) and B (of size $n \times p$) can be multiplied to produce a new $m \times p$ matrix $C = A \cdot B$. The element c_{ij} in the resulting matrix is given by:

$$c_{ij} = \sum_{i=1}^{n} a_{ik} b_{kj}$$

or in words, the rows are multiplied of A are multiplied with columns of B and provide the resulting element at the index combination where the row and column intersect.

3.4 Matrix Multiplication is not Commutative

The order of a matrix product is important, and result in different results. For the previously defined two matrices, A and B, we obtain:

$$AB = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$
$$BA = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

and we see that the results are not identical, confirming that the order matters, and we say that that matrices generally do not commute.

3.5 Matrix Multiplication is Associative

Matrix multiplication is associative:

$$(AB)\,C = A\,(BC)$$

for example, using the previously defined matrices, A and B, together with matrix

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

we can confirm that

$$(AB)\,C = A\,(BC) = \begin{pmatrix} 41 & 25 \\ 93 & 57 \end{pmatrix}$$

3.6 Matrix Transpose

Transposing a Matrix: The transpose of a matrix A, denoted A^T, is obtained by switching its rows with its columns. Mathematically:

$$\left(A^T\right)_{ij} = a_{ji}$$

that is row and column indices are exchanged, or equivalently, upper and lower triangular matrix are exchanged.

4 Special Types of Matrices

4.1 Identity Matrix

Definition: The identity matrix I is a square matrix in which all the elements on the main diagonal are 1 and all other elements are 0:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Multiplying any matrix by I (on the appropriate side) leaves the original matrix unchanged.

4.2 Diagonal Matrix

A diagonal matrix has nonzero entries only on its main diagonal:

$$I = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

4.3 Symmetric Matrix

A matrix A is symmetric if it equals its transpose, i.e., $A = A^T$. Symmetric matrices appear in many physical contexts, such as in describing moments of inertia or in the metric tensors in general relativity. That is the upper and lower triangular matrices are identical. Here is an example of a symmetric matrix:

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

5 Exercises

1. Add the following two matrices: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$.
2. Multiply the matrices from the previous exercise.
3. Using the matrices from exercise 1, show that the order in a matrix product is important: $A \cdot B \neq B \cdot A$.

4. Defining an additional matrix

$$C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

show that matrix multiplication is associative: $(AB)\,C = A\,(BC)$

5. Provide an example, where the order of the two matrices in a matrix product does not matter.

6. Using the three matrices, A, B and C, show that matrix multiplication is associative.