Quantum Mechanics

August 21, 2015

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Calculate the scattering amplitude and the total scattering cross section for the potential

$$V(\vec{r}) = V_0 \frac{e^{-ar}}{r}$$

in the first Born approximation. What happens in the limit $a \to 0$?

Note: The magnitude of the momentum transfer $q = |\vec{k} - \vec{k}'|$, where $\vec{k}, \vec{k}'$ are the in- and outgoing momenta, is related to the scattering angle θ via $q = 2k \sin(\theta/2)$.

Problem 2

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them $|\uparrow\rangle$ and $|\downarrow\rangle$) representing the possible paths taken by the particles. A particle emerging in state $|\uparrow\rangle$ produces a wave function at the screen of the form $\psi_\uparrow(x)$ (where x is a coordinate along the screen) while a particle emerging in state $|\downarrow\rangle$ produces a wave function $\psi_\downarrow(x)$. The evolution from the wall with the slits to the screen is linear in the input state. As the source repeatedly spits out particles, the screen counts how many particles hit at each location x. Suppose, for simplicity, that $\psi_\uparrow(x) = \exp(ik_\uparrow x)$, $\psi_\downarrow(x) = \exp(ik_\downarrow x)$, where $k_\uparrow$, $k_\downarrow$ are some real constants.

a) If the particles are all spat out in the state $|\uparrow\rangle$, what is the x-dependence of the resulting pattern $P_\uparrow(x)$? Show a graphical plot of this distribution on the screen.

b) If the particles are all spat out in the (normalized) state $|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle$, what is the x-dependence of the resulting pattern, $P_\psi(x)$? Assume μ and λ are real-valued.

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space Ω_Ξ. Suppose these interactions are described by the Hamiltonian

$$H = \sigma^z \otimes M$$

acting on $\Omega_2 \otimes \Omega_\Xi$, where $\sigma^z = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ acts on the Hilbert space Ω_2 of particle paths and M is an operator acting on the Hilbert space of the environment. Suppose the initial state of the whole system is

$$|\psi_0\rangle = (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_\Xi \ ,$$

and that

$$M = m\sigma^x = m\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)_\Xi$$

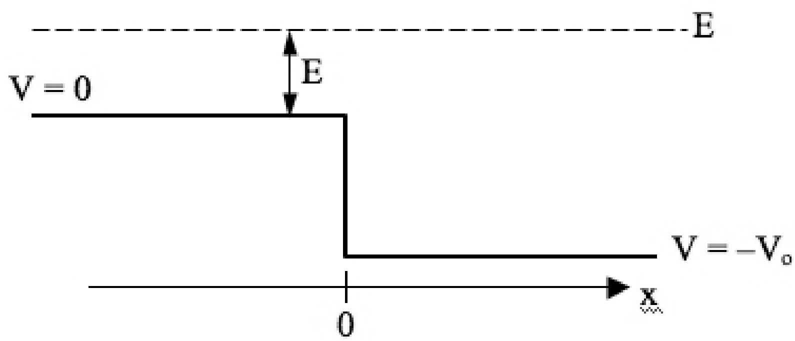
c) Find $|\psi(t)\rangle$, the state of the whole system at time t.

d) How does the interference pattern depend on x and t? For simplicity, consider the case where $\mu = \lambda = 1/\sqrt{2}$.

e) Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.

f) What would happen if instead the initial state of the environment were an eigenvector of M?

Problem 3



Consider a "downstep" potential, which drops at $x = 0$ as one goes from left to right:

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ -V_0 & \text{for } x > 0 \end{cases}$$

where $V_0 > 0$.

a) In classical physics, consider a particle of mass m coming in from the left with initial velocity v_i. What happens to it at the downstep? Find the total energy in terms of v_i and the final velocity v_f in terms of m, v_i and V_0.

b) In quantum physics, solve the time-independent Schrödinger equation (TISE) for fixed energy $E > 0$ in both regions and impose appropriate boundary conditions at $x = 0$; write down the equations for these boundary conditions.

c) Interpret these solutions as a steady flux of particles, and assume there are no particles coming in from the right. Calculate the reflection coefficient R and transmission coefficient T and write them in terms of E and V_0. Discuss how your answer compares to the classical result.

Problem 4

Consider an electron moving in the x-y plane with momentum $\vec{p}$ under the influence of a uniform magnetic field $\vec{B}$ in the z-direction: $\vec{B} = B\hat{k}$. The electron has a magnetic moment given by

$$\vec{\mu} = g\frac{e}{2m}\vec{S},$$

where e and m are the electron charge and mass, $\vec{S}$ its spin, and the gyromagnetic ratio g is not exactly 2, as predicted by the Dirac theory, but rather $g = 2(1+a)$, where a is a small number that has been measured extremely precisely. You have learned that the magnetic field causes the electron to move in a circle with "cyclotron frequency" $\omega = eB/m$ (you don't need to show this here); you will show that the small deviation of g from 2 (*i.e.*, the fact that $a \neq 0$) causes the electron spin to "precess" at slightly different frequency and this can be exploited to measure this quantity.

a) Write the Hamiltonian $\mathcal{H}$ for the electron in the magnetic field. Hint: The gauge-invariant velocity operator is

$$\vec{v} = \frac{1}{m}\left(\vec{p} - e\vec{A}\right),$$

where $\vec{A}$ is the electromagnetic vector potential. (1 point)

b) Show that the choice of

$$\vec{A} = (0,\, Bx,\, 0)$$

produces the right magnetic field for this problem. (1 point)

c) Compute the commutators $[v_i, \mathcal{H}]$ of the components of $\vec{v}$ with the Hamiltonian; feel free to use the choice of gauge from b). (2 points)

d) Consider the operators

$$\mathcal{O}_1 = S_x v_x + S_y v_y$$

and

$$\mathcal{O}_2 = S_x v_y - S_y v_x$$

and their expectation values

$$C_i(t) = \langle \mathcal{O}_i \rangle\,.$$

Using the Ehrenfest theorem, derive the two (coupled) differential equations that describe the time evolution of $C_i(t)$. (4 points)

e) A beam of electrons, initially in a state with known values $C_1(0)$ and $C_2(0)$ and with velocity $\vec{v}$ enters a region of magnetic field $\vec{B}$ as described above. Notice that one of the two operators in the previous step describes the projection of the spin vector onto the velocity vector. Combining the two equations you found in d), derive a differential equation for the time dependence of the expectation value of this operator and solve to show how the angle between spin and velocity changes with time. Explain how a measurement of this (periodic) motion can be used to measure the deviation of g from the Dirac value. Verify that, if $g = 2$ exactly, then the spin always follows the movement of the velocity vector.

(2 points)

Problem 5

An electron moving in a conjugated bond framework may be regarded as a particle in a box. An externally applied electric field of strength ε interacts with the electron in a fashion described by the perturbation

$$V = e\varepsilon \left(x - \frac{L}{2} \right)$$

where x is the position of the electron within the conjugated bond system, e is the electron charge, and L is the length of the conjugated bond system.

a) Evaluate the first-order correction to the energy of the ground state and the first-order wavefunction for the ground state. Make a rough sketch of $\psi^{(0)}_{grd.st.} + \psi^{(1)}_{grd.st.}$ as a function of x AND physically interpret the graph.

b) Using your answer from a) compute the induced dipole moment caused by the polarization of the electron density due to the applied electric field effect

$$\mu_{induced} = -e \int dx\, \psi^* \left(x - \frac{L}{2} \right) \psi$$

c) Determine the polarizability, α, of the electron in the ground state of the conjugated bond framework and explain physically why α should depend as it does upon the length of the conjugated bond framework, L.