Quantum Mechanics

August 22, 2014

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider a particle of mass, m, moving in one dimension, x, in a potential defined by

$$V(x) = \infty \qquad \text{for } x \leq 0$$
$$V(x) = \frac{\hbar^2}{2m}\beta x \quad \text{for } x > 0$$

For the purposes of this problem suppose you wish to make a variational estimate of the ground-state energy of the system.

a) Using dimensional analysis write down a formula for the result of the variational energy within an unknown dimensionless constant.

b) Calculate the variational estimate of the ground state energy using a trial wave function proportional to xe^{-ax}.

c) Calculate the variational estimate of the ground state energy using a trial wave function proportional to xe^{-bx^2}.

d) Which of the two estimates is closer to the true value.

Problem 2

Show that the energies of the bound states of a system with reduced mass μ decrease with increasing μ, independently of the form of the potential.

Hints: Consider a Hamiltonian $\mathcal{H}(\lambda)$ which depends on some parameter λ. Starting from the Schrödinger equation, $\mathcal{H}\,|\Psi\rangle = E\,|\Psi\rangle$ show that the eigenstate energies E_i depend on λ such that

$$\frac{\partial E_i}{\partial \lambda} = \left\langle \frac{\partial \mathcal{H}}{\partial \lambda} \right\rangle$$

and apply this to the case where the parameter $\lambda = \mu$, the reduced mass.

Problem 3

A particle of mass m and charge q sits in a 3D harmonic potential $V = \frac{1}{2}k(x^2 + y^2 + z^2)$. At time $t = -\infty$ the oscillator is in its ground state. It is then perturbed by a spatially uniform time-dependent electric field

$$\mathbf{E}(t) = Ae^{-t^2/\tau^2}\,\hat{z} \tag{1}$$

where A and τ are constant. Using the lowest (non-vanishing) order perturbation theory, calculate the probability that the oscillator is in an excited state at $t = \infty$.

Problem 4

Consider a two-level system described by a Hamiltonian H with $H|i\rangle = E_i|i\rangle$, $i = 1, 2$. For simplicity, let $E_1 = 0$, $E_2 = \epsilon$. The system can furthermore decay, where the details of the decay mechanism are immaterial; the decay rates Γ_i of the two eigenstates are known. Now, introduce an interaction Hamiltonian H' with nonzero matrix elements $\langle 1|H'|2\rangle = \langle 2|H'|1\rangle = B$. Given a large number N of such systems, subject to the full Hamiltonian $H + H'$, which initially, at $t = 0$, are all in the state $|2\rangle$, derive the number of systems which will have decayed as a function of time t.

Problem 5

1. Consider a Hamiltonian of the form

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V_1(x^2+z^2) + V_2(y^2),$$

where $V_1 \neq V_2$ are not further specified. Name two nontrivial operators that commute with H (nontrivial in the sense that not a constant nor H itself not $\frac{\partial}{\partial t}$. Also the two operators should not just be multiple of each other). Explain your answer. (2 points)

2. Consider two identical fermions with the same spin quantum number. Is

$$\psi(\vec{r}_1,\vec{r}_2) = (y_1 - y_2)\left[f(\vec{r}_1)g(\vec{r}_2) + f(\vec{r}_2)g(\vec{r}_1)\right]$$

an allowed wave function for this system? Here $\vec{r}_1$ and $\vec{r}_2$ denote the position of the two fermions and f and g are not further specified. Explain your answer.(2 points)

3. Consider a single-particle Hamiltonian of the form

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r}^{\,2}),$$

where V is not further specified except that it has such a shape that the Hamiltonian has several bound state solutions. Which of the following statements are true, regardless of the shape of V (3 points)

   a.) The ground state is always an s-state ($l=0$)
   b.) The 1^{st} excited state is always an s state ($l=0$)
   c.) The 1^{st} excited state is always a p state ($l=1$)
   d.) The 1^{st} excited state is either an s state ($l=0$) or a p state ($l=1$) and both have the same energy
   e.) The 1^{st} excited state always has a node (goes through zero) either as a function of the radial variable or of the angular variables
   f.) The 1^{st} excited state is never a d state ($l=2$)

4. Consider two identical fermions in one dimension (with identical spin quantum numbers) each with mass m bound together by a harmonic oscillator potential

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}\left(x_1 - x_2\right)^2.$$

What is the energy of the ground state? (3 points)