Quantum Mechanics

August 23, 2013

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Consider the non-relativistic interaction of a particle of mass, m, and energy, E, with a three dimensional potential well defined by

$$V(r) = -V_0 \quad \text{for } r \leq R; \quad V(r) = 0 \quad \text{for } \ r > R.$$

For notation use

$$K_0 = \sqrt{2mV_0}/\hbar; \quad k = \sqrt{2mE}/\hbar$$

Consider only the s-wave interaction.
1) What is the condition for a bound state in the well? Do not attempt to solve for the energies.
2) What are the values of V_0 which permit a bound state at zero energy?
3) For positive values of E find an expression for the s-wave phase shift, δ, where the external wave function can be written as

$$C(e^{2i\delta}e^{ikr} - e^{-ikr})/r.$$

4) Show that for low energy $\delta = ak$ where a is constant (the scattering length). Find the constant a.
5) Show that a is the value of the scattering amplitude at zero energy where the full scattering amplitude is given by

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(e^{2i\delta}-1)P_\ell(\cos\theta)$$

and $P_\ell(x)$ is the Legendre polynomial.
6) Sketch the behavior of a as a function of K_0 up to 6 (take R=1 for your sketch).

Problem 2

Let $a^\dagger$, a denote the raising and lowering operators of a harmonic oscillator with frequency ω.

a.) Consider the Hamiltonian

$$H = \hbar\omega(a^\dagger a + 1/2) + \lambda a^\dagger a^\dagger a^\dagger aaa \ .$$

Give the spectrum of H.

b.) Consider the Hamiltonian

$$H = \hbar\omega(a^\dagger a + 1/2) + \lambda(a^\dagger a^\dagger a^\dagger a^\dagger + aaaa) \ .$$

Taking λ to be a small parameter in this case, give the spectrum of H to second order in λ.

Problem 3

Starting from the product basis, construct an orthonormal coupled basis (labeled by well-defined total spin squared and z-component of total spin) of the space of spin states of three spin-1/2 objects. Consider the objects to be distinguishable.

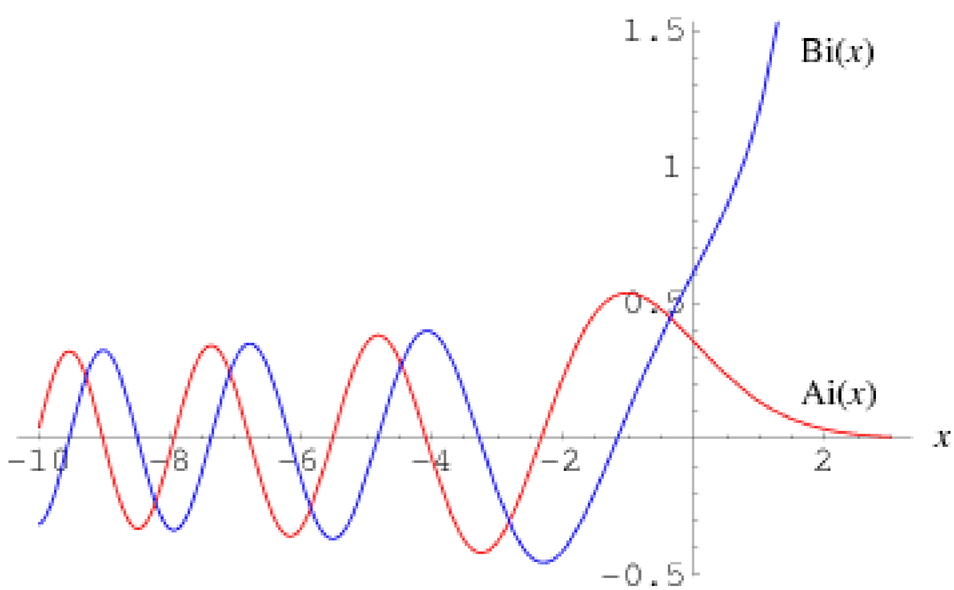
Problem 4

A neutron is located above a perfectly reflective surface at $z = 0$, with only the earth's gravitational field acting on it.

1. Write the Schrödinger equation with the potential clearly defined.(2 points)
2. Show, by doing a variable transformation, that this is the Airy equation. (3 points)
3. Make use of the boundary conditions to demonstrate that the energy eigenvalues are quantized and express these eigenvalues in terms of λ_n, the zeros of the Airy function: $\mathrm{Ai}(\lambda_n) = 0$; calculate the ground-state energy in eV. . . . . (4 points)
4. Is the second Airy function, $\mathrm{Bi}(x)$, also an acceptable solution? Explain. (1 point)

<u>Notes:</u>

1. The Airy equation is $\psi'' - x\psi = 0$ with the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ as solutions; these are shown in the figure below
2. The first zero of the Airy function $\mathrm{Ai}(x)$ is $\lambda_1 \simeq -2.338$
3. The neutron mass is $m \simeq 940\,\mathrm{MeV}/c^2$
4. $\hbar = 1.054 \times 10^{-34} m^2 kg/s$
5. Take the speed of light to be $3 \times 10^8 m/s$
6. A useful combination of constants is $\hbar c \simeq 197\,\mathrm{MeV\,fm}$



Problem 5

1. An electron spin in a magnetic field is described by the Hamiltonian

$$H = \mu_B \vec{B} \cdot \vec{\sigma}$$

where σ_i are the Pauli matrices. At time $t = 0$ the electron is in the state $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ when a magnetic field $\vec{B} = B_0 \vec{e}_x$ is applied, where B_0 is constant.

   a. Write down the time-dependent Schrödinger equation $i\hbar \frac{\partial}{\partial t}\psi = H\psi$ for the spinor $\psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$ (1 point)

   b. Calculate $\psi(t)$ at time t by solving the time dependent Schrödinger equation with initial condition $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (2 points)

   c. Use this result to calculate (2 points)

$$\langle S_z(t) \rangle \equiv \frac{1}{2}\psi^\dagger(t)\sigma_z\psi(t)$$

2. Two particles with the same mass m are bound by a harmonic oscillator potential in 3 dimensions

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}_1^2 - \frac{\hbar^2}{2m}\vec{\nabla}_2^2 + \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2$$

   a. Write down the Hamiltonian in the center of mass frame. (1 point) From this point on, consider the Hamiltonian in the center of mass frame.

   b. If the two particles are indistinguishable particles with spin zero, what is the energy of the two lowest allowed states? (2 points)

   c. If the two particles are indistinguishable particles with spin zero, what is the degeneracy of the first excited state? (2 points)