Classical Mechanics

August 28, 2013

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and put your name on each sheet.

Problem 1

A railroad flatcar consists of a platform of mass M plus four wheels, each wheel being a uniform solid disk of mass m and radius a. Initially the flatcar is rolling freely along a horizontal track with speed v_i (nothing is pulling the flatcar). It passes a point where a mass μ is dropped vertically onto the flatcar. Find the subsequent speed of the flatcar, v_f (the wheels never slip).

Problem 2

A simple model for a molecule is to place six frictionless beads of equal mass on a wire string. The beads are connected by equal springs with a spring constant K. The wire is fixed in space and atoms can only move along the wire. Positive displacements along the wire are counted clockwise.

Some hints on the eigenvalue spectrum and eigenvectors of circulant matrices (no need to prove):

$$C = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & & & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & & c_2 \\ c_2 & c_1 & c_0 & \cdots & & c_3 \\ \vdots & & \vdots & & \vdots & \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 \end{pmatrix}$$

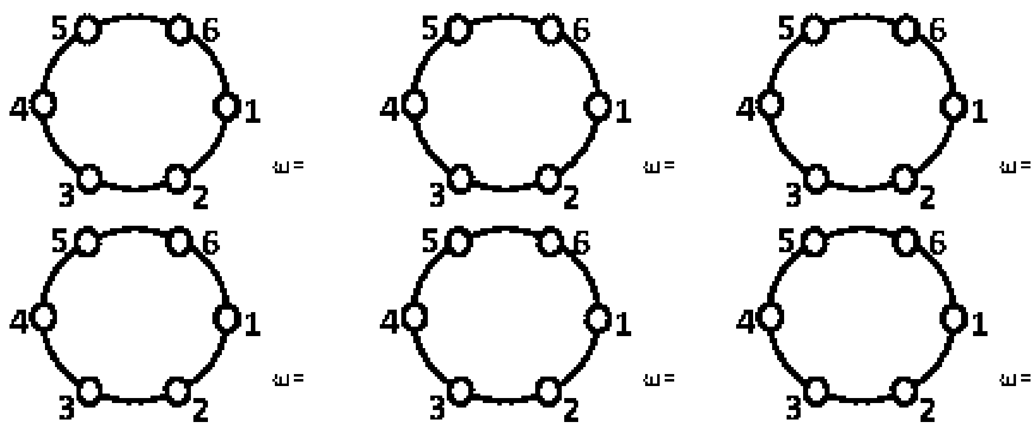
The eigenvalues of a circulant ($Cv = \lambda v$) matrix are given by:
($\overline{\omega} = e^{\frac{i2\pi j}{n}}$ with $j = 0, 1, \ldots\ n-1$ and $i = \sqrt{-1}$

$$\lambda_j = c_0 + c_{n-1}\overline{\omega}_j + c_{n-2}\overline{\omega}_j^2 + \ldots + c_1\overline{\omega}_j^{n-1}$$

And the corresponding eigenvectors:

$$v_j = (1, \overline{\omega_j}, \overline{\omega_j}^2, \ldots, \overline{\omega_j}^{n-1})^T$$

a) The motion of the atoms is conveniently parametrized in terms of the position along the circle. Derive or write down the kinetic energy for one atom in terms of the arc length s.
b) Derive or write down the equations of motion.
c) Calculate or write down the eigenfrequencies of the normal modes, indicating any degeneracies. In the pictures below indicate the direction of motion by drawing an arrow near each mass indicating the direction of motion and shading the atoms that are at rest.



d) If you find one or more "vibrations" with *zero* frequency, explain their origin.
e) For which normal modes does the center of mass oscillate? Which modes could be related to the vibrational modes of a real molecule?
f) Does the highest frequency depend on the number of atoms on the wire? Explain.

Problem 3

A heavy particle slides down a curve in the vertical plane starting from the point $P = (a, A)$. Find the curves such that the particle reaches the vertical line $x = b$, in the shortest time.