

# Statistical Mechanics

August 29, 2012

Work 2 (and only 2) of the 3 problems. Please put each problem solution on a separate sheet of paper and put your name on each sheet.

## Problem 1

Assume that the tension of a rubber band is given as

$$F_T = (\alpha + \beta T)(l - l_0)$$

where  $T$  is the temperature,  $l$  is the length and  $\alpha$ ,  $\beta$ , and  $l_0$  can be regarded as constants, all of which are positive. Consider that the rubber band is already stretched to length  $l_1$  and it is at the equilibrium temperature  $T_1$ . Then, the rubber band undergoes the following 4-step process:

Leg 1: The rubber band is stretched adiabatically from  $l_1$  to  $l_2$  (thereby increasing the temperature to  $T_2$ ).

Leg 2: The rubber band returns to the temperature  $T_1$  while keeping the length at  $l_2$ .

Leg 3: The rubber band then contracts adiabatically from  $l_2$  to  $l_1$  (thereby decreasing the temperature to  $T_0$ ).

Leg 4: Finally, the rubber band returns to its original temperature  $T_1$  at length  $l_1$ .

The volume,  $V$ , is constant during adiabatic expansion and contraction (the band becomes thinner as it is being stretched adiabatically), and thus we can assume that the change in internal energy

$$dU = -C_V dT,$$

where  $C_V$  is the heat capacity at constant volume. The system exchanges heat with the environment only during the  $2 \rightarrow 3$  and the  $4 \rightarrow 1$  legs of the cycle. Compute these heat exchanges. For which of these is the magnitude of the exchange greater? In other words, is the complete cycle a heat engine or a refrigerator? Hint: Show that the tensions in the adiabatic portions of the cycle can be expressed as,  $F_T = M(l)(l - l_0)$  where  $M(l)$  is a function of  $l$ . Determine this function and use it in the computation of the non-adiabatic contributions.

## Problem 2

An ideal gas is a system in which the mutual potential energies of the molecules comprising the gas may be neglected owing to their large mean free path. Deviations from the ideal gas equation of state  $PV = NkT$  are observed as the gas is compressed or the molecules making up the gas increase in size and complexity. Consider a gas of  $N$  identical molecules in a volume  $V$  subject to a short-range two-body central potential  $U(r)$ . We can assume that the molecules have no excited internal degrees of freedom and, since the behavior of the gas is not due to quantized energy levels, classical methods may be employed. Starting therefore from the classical phase space integral for the partition function, show that the leading correction to the ideal gas equation of state is

$$PV = NkT \left( 1 - \frac{\beta}{2v} \right),$$

where  $v$  is the volume per particle and the constant  $\beta$  is defined by

$$\beta = 4\pi \int_0^\infty r^2 (e^{-U(r)/kT} - 1) dr .$$

Hint: The quantities

$$f_{ij} = \exp(-U(r_{ij})/kT) - 1 ,$$

where  $r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$ , are very small in most of phase space when the system is dilute (as an additional advantage, they remain finite in the interaction region even when the potential contains a hard core, as is the case in many applications).

### Problem 3

Consider a system of  $N$  independent quantum-mechanical harmonic oscillators with the energy levels

$$\epsilon_n = \hbar\omega_0 \left( n + \frac{1}{2} \right). \quad (1)$$

Find the canonical partition function, Helmholtz free energy, average energy, and specific heat of this system. Sketch the behavior of the specific heat as a function of temperature.