Quantum Mechanics

September 26, 2011

Work 4 (and only 4) of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

1. For a negative charge bound by a heavy positive charge, the following observables are conserved and can be determined simultaneously: energy, total angular momentum, \( L_z \), parity.

   a.) suppose one applies a homogenous magnetic field in the \( \hat{z} \)-direction, which of the above observables are no longer conserved?

   b.) suppose one applies a homogenous electric field in the \( \hat{z} \)-direction, which of the above observables are no longer conserved? (ignore tunneling effects)

      \[ \text{(3 points)} \]

2. Two protons with spin parallel are bound by a harmonic oscillator potential in 3 dimensions

   \[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2} k (\vec{r}_1 - \vec{r}_2)^2. \]

   a.) What is the ground state energy? What is the energy spectrum? Note: you must be very specific about the range for any parameters appearing in your answer! Explain how you obtained your answer!

      \[ \text{(5 points)} \]

   c.) What is the degeneracy of the ground state? Explain how you obtained your answer!

      \[ \text{(2 points)} \]
Problem 2

A particle is in a three-dimensional box (infinite square well) with all three sides $L$. If the particle is in the first excited state, what can you tell about its angular momentum?
Problem 3

Consider a bound state solution, $\psi(r)$, of the Schrödinger equation in three dimensions with a spherically symmetric potential. Show that the energy, $E$ is given by

$$E = \frac{\int dr V(r) \psi(r)}{\int dr |\psi(r)|^2}$$

(note that the integral is over the wave function, not the square of the wave function).

For the case of

$$V(r) = \frac{2\hbar^2}{m} \lambda^2 r^2$$

show explicitly that this relation gives the correct energy by solving for the ground state wave function and energy and comparing with this formula.
Problem 4

Consider a Hamiltonian of the form

\[ H = H^0 + H'(t) \]

where \( H^0 u_n(x) = E_n u_n(x) \) and \( H' \) is to be considered as a perturbation. The full wave function can be expanded in terms of the eigenstates of \( H^0 \)

\[ \psi(x,t) = \sum_m a_m(t) e^{-iE_m t/\hbar} u_m(x) \]

where we will take \( \langle m|m' \rangle = \int u_m^* u_m' dx = \delta_{m,m'} \)

1) Given the initial conditions \( a_n(0) = 1 \) and \( a_k(0) = 0 \) for \( k \neq n \) show that.

\[ a_k(t) = -i \frac{t}{\hbar} \int_0^t \langle k|H'(t)|m> e^{i\omega_m t'} dt' \]

where \( \omega_{kn} = (E_k - E_n)/\hbar \). Assume that (for the range of times being considered) \( a_k(t) \ll 1 \) for \( k \neq n \)

2) If \( H'(t) = 2V_0 \cos \omega t \) show that the probability of finding the system in the state \( f \) at time \( t \), given that it was in the state \( i \) at \( t = 0 \), is:

\[ P_f(t) = |a_f|^2 = \frac{|V_{fi}|^2}{\hbar^2} \left( \frac{\sin^2 \delta}{\delta} \right) t^2 \]

where \( V_{fi} = \langle f|V_0|i > \) and \( \delta = (\omega_{fi} - \omega)t/2 \).

3) If the transition is to a number of states around some energy \( E_f \) where the density of states is \( N(E_f) \) show that the transition rate is

\[ W_{fi} = \frac{P_f(t)}{t} = \frac{2\pi}{\hbar} |V_{fi}|^2 n(\hbar\omega_{fi}) \]

This last formula is known as “Fermi’s Golden Rule”.
Problem 5

a) Set up Schrödinger’s equation for the He atom and use the variational method to approximate the energy of the ground state. In this calculation take

\[ u = Ae^{-c(r_1+r_2)} \]

for the trial function with \( c > 0 \) as a parameter to be determined by variation.

b) Confirm that your result is, in fact, a minimum.

c) The experimental value for the binding energy ground state of the He atom is -79.0 eV. How does this compare with your answer and how can you explain any discrepancy?