Problem 1

Consider a degenerate Fermi gas containing \( n \) particles per unit volume. Let \( \epsilon_F \) be the Fermi energy of this gas.

(a) Derive an expression for the isothermal compressibility, \( \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \), of this gas at zero temperature.

(b) Derive an expression for the thermal expansion coefficient, \( \alpha = \frac{1}{3V} \left( \frac{\partial V}{\partial T} \right)_P \), of this gas. 
(Hint: Use the chain rule for \( V, P, \) and \( T \).)
Problem 2

Paramagnetism of a stationary particle having charge $q$, mass $m$, and spin $\frac{1}{2}$, in a constant uniform magnetic field $B$.

a) The conventional treatment assumes a magnetic moment $\mu = (gq/2mc)S$, where $g$ is the “$g$-factor” ($g \approx 2$ for electrons), $S$ is the spin angular momentum operator, with eigenvalues $\pm \frac{1}{2}\hbar$ for $S_z$, so that the eigenvalues of $\mu_z$ are

$$\mu_z = \pm \frac{gq}{4mc}\hbar \equiv \pm \mu_B$$

where $\mu_B$ is the Bohr magneton.

Let $H_P$ be the Pauli Hamiltonian, $H_P = -\mu \cdot B = -\Omega \cdot S$ where $\Omega = gqB/2mc$. The eigenvalues of $H_P$ are $E_\pm = \mp \frac{\hbar}{2}\Omega_z$ ($B$ in the $z$-direction).

i) What is the (single particle) partition function $Z$ for temperature $T$?

ii) The expectation $<\mu_z>$ is defined by

$$<\mu_z> = Z^{-1}[\mu_+e^{-\beta E_+} + \mu_-e^{-\beta E_-}]$$

where $\beta \equiv (k_BT)^{-1}$ and $k_B$ is Boltzmann’s constant. Show that

$$<\mu_z> = \mu_B\tanh(\frac{\beta \hbar \Omega_z}{2})$$

b) In the 1970’s, R. Young investigated quantization of a spinning extended charged particle. For a spherically symmetric particle whose center is at rest, with moment of inertia $I$ about its center, he showed that the Hamiltonian and the magnetic moment are

$$H = \frac{1}{2I}(S - I\Omega)^2; \quad \mu = (gq/2mc)(S - I\Omega)$$

i) Using appropriate eigenvalues of $\mu_z$ and $H$ in Eq. (2), show that now

$$<\mu_z> = \mu_B[\tanh(\frac{\beta \hbar \Omega_z}{2}) - 2I\Omega_z/\hbar]$$

ii) Consider room temperature, $k_BT = \frac{1}{30}$ eV, a large field $B_z = 10^4$ Gauss, and proton parameters, $q = e = 1.6 \times 10^{-12}$ esu, $mc^2 = 931$ MeV $\rightarrow m = 1.6 \times 10^{-24}$ g, and $I = ma^2$ with $a = 10^{-12}$ cm. Show that then $\beta \hbar \Omega_z/2 << 1$, so that $<\mu_z> = \frac{1}{2}\mu_B\beta \hbar \Omega_z[1 - 4I/\beta \hbar^2]$, and $(4I/\beta \hbar^2) \approx 2.5 \times 10^{-7}$ which is certainly negligible. (1 eV = $1.6 \times 10^{-12}$ erg, $c = 3 \times 10^{10}$ cm/s.)
Problem 3

A rotator with principal moments of inertia $A$, $B$ and $C$ is described by the Hamiltonian

$$H = \frac{1}{2A\sin^2 \theta} ((p_\phi - p_\psi \cos \theta) \cos \psi - p_\theta \sin \theta \sin \psi)^2$$

$$+ \frac{1}{2B\sin^2 \theta} ((p_\phi - p_\psi \cos \phi) \sin \psi + p_\phi \sin \theta \cos \psi)^2 + \frac{1}{2C} p_\psi^2$$

where $\theta$, $\phi$ and $\psi$ are the Euler angles.

a. Derive the rotational partition function for this polyatomic molecule in the classical approximation, up to a constant. Hint: Carry out the integrations in the order $p_\theta$, $p_\phi$, $p_\psi$, 

b. Derive the specific heat of this rotator; interpret your answer.