Electromagnetism

August 31, 2009

Work 4 of the 5 problems. Please put each problem solution on a separate sheet of paper and your name on each sheet.

Problem 1

Two infinitely long wires both run parallel to the x-axis. One wire has charge density $+\lambda$ and is located at $(y, z) = (a, 0)$; the other has charge density $-\lambda$ and is located at $(y, z) = (-a, 0)$. For definiteness give the wires arbitrarily small, and equal, diameters ϵ

a. Find the electric potential difference between the wires.

b. Find the capacitance per unit length of the two-wire system.

c. Find an expression for the electric potential $V(x, y, z)$ at any point.

d. Find an equation for $z(y)$ describing an equipotential surface with potential V_0.

Problem 2

The Drude theory treats electrical transport in terms of free electrons that undergo collisions. It results in an expression for the electrical conductivity:

$$\sigma = (ne^2\tau)/m \ ,$$

where n is the number of electrons, e is the electron charge, τ is the average time between collisions (also called relaxation rate) and m is the mass of the electrons. Strictly speaking, the above equation applies only to free electrons. In metals, however, electrons occupy electron bands, and they behave as if they were almost free. Electron correlations can then be accounted for by replacing m with an "effective mass" m^*.
Now, consider a ferromagnetic metal, in which the current is carried by electrons in *free-electron-like* conduction bands containing equal numbers n of spin-up ($\uparrow$) and spin-down ($\downarrow$) electrons. The electron distributions for spin-up, spin-down and filled bands are denoted by $f_\uparrow$, $f_\downarrow$ and f_0, respectively. The cross sections for collisions are different for spin-up and spin-down electrons, and the corresponding relaxation rates are $\tau_\uparrow$ and $\tau_\downarrow$. Moreover, there is the possibility of spin-flip scattering with a relaxation rate $\tau_{\uparrow\downarrow}$.
This then leads to the following coupled Boltzmann transport equations for the electron distributions of the $\uparrow$ and $\downarrow$ electrons:

$$\begin{aligned}
-e\vec{E}\cdot\vec{v}\,\frac{\partial f_0}{\partial E} &= \left\{f_\uparrow(\vec{k}) - f_0(\vec{k})\right\}\Big/\,\tau_\uparrow \;+\; \left\{f_\uparrow(\vec{k}) - f_\downarrow(\vec{k})\right\}\Big/\,\tau_{\uparrow\downarrow} \qquad \text{for } \uparrow \ , \\
-e\vec{E}\cdot\vec{v}\,\frac{\partial f_0}{\partial E} &= \left\{f_\downarrow(\vec{k}) - f_0(\vec{k})\right\}\Big/\,\tau_\downarrow \;+\; \left\{f_\downarrow(\vec{k}) - f_\uparrow(\vec{k})\right\}\Big/\,\tau_{\uparrow\downarrow} \qquad \text{for } \downarrow \ ,
\end{aligned}$$

where $\vec{E}$ the electrical field, the various τs are the relaxation rates and the various fs are the electron distribution functions of the respective spin configurations.

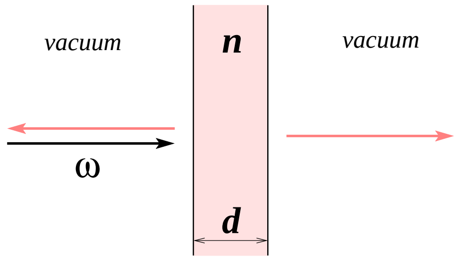
a. Find a set of equations with decoupled electron distribution functions $f_\downarrow(\vec{k})$ and $f_\uparrow(\vec{k})$.

b. Find the effective scattering rates for spin $\uparrow$ and spin $\downarrow$ electrons and, using $\sigma = (ne^2\tau)/m^*$ for the respective spin configurations, determine the expressions for $\sigma_{\uparrow,\text{eff}}$ and $\sigma_{\downarrow,\text{eff}}$.

c. Find an expression for the total resistance $\rho = \sigma^{-1} = (\sigma_{\uparrow,\text{eff}} + \sigma_{\downarrow,\text{eff}})^{-1}$ of a ferromagnetic metal in terms of $\rho_\uparrow$, $\rho_\downarrow$ and $\rho_{\uparrow\downarrow}$. Using Kirchhoff's laws, draw an equivalent resistor circuit with resistors $\rho_\uparrow/2$, $\rho_\downarrow/2$ and $\rho_{\uparrow\downarrow}$.

Problem 3

A plane electromagnetic wave of frequency ω is incident normally from vacuum on a slab of dielectric material of thickness d with a refraction index $n > 1$. Find the reflection and transmission coefficients in terms of n, ω, and d, assuming that the magnetic permeability of the material is $\mu = \mu_0$. Under what conditions would you expect the reflection coefficient to be zero?



Problem 4

Consider a pure electric monopole particle, charge q, mass m, moving in the $x-y$ plane only, in a constant uniform magnetic field in the $+z$ direction, $\mathbf{B} = \mathbf{e}_z B,\ B > 0$. The particle has initial velocity $\mathbf{v}(t=0) = \mathbf{e}_y v_0,\ v_0 > 0$, and initial position $y(0) = 0,\ x(0) = -v_0/\Omega$. No other external forces are applied.

a) If the radiation reaction (RR) force is neglected, show that according to Newton's second law, $\dot{\mathbf{v}} = -\mathbf{\Omega} \times \mathbf{v}$, the particle moves in a circle of radius $r_c = v_0/\Omega$ at speed v_0, so that its energy is conserved. Here $\mathbf{\Omega} = \mathbf{e}_z \Omega$, $\Omega \equiv qB/mc$ (Gaussian). Note: Ω is called the cyclotron frequency. Solve for $\dot{x}(t),\ \dot{y}(t),\ x(t),\ y(t)$ to show this clearly.

b) We know that, classically, an accelerated charged particle radiates energy, so for this system there *must* be a RR force $\mathbf{F}^{RR}$ in the motion equation. A reasonable approximation is $\mathbf{F}^{RR} \approx m\Omega^2\tau\mathbf{v}$, where $\tau \equiv 2q^2/3mc^3$ is the RR time constant, so that the motion equation is

$$\dot{\mathbf{v}} = -\mathbf{\Omega} \times \mathbf{v} - \Omega^2\tau\mathbf{v}.$$

Solve this equation for $\dot{x}(t),\ \dot{y}(t)$, for the conditions given above.

c) Find expressions for $E(t)$, the energy as a function of time; for $T_{\frac{1}{2}}$, the time required to lose half the initial energy; and for $N_{\frac{1}{2}}$, the number of orbits executed during this time interval.

d) Then, put in the following "realistic" numerical values:
$q = e = 4.8 \times 10^{-10}$ statC; $m \approx m_e \approx 10^{-27}$ g; $c = 3 \times 10^{10}$ cm s^{-1}; $B = 10^4$ Gauss; $v_0 = 10^7$ cm s^{-1} $\approx$ thermal speed of an electron at room temperature. Then calculate values for $\Omega,\ r_c,\ N_{\frac{1}{2}},\ T_{\frac{1}{2}}$. [Note; Just plug the given numerical values into the formulas, and everything will come out OK in cgs (Gaussian) units, because the formula for Ω is a Gaussian formula.]

Problem 5

Laser radiation is unique, but many features may be understood using a simple classical model. The scalar electric field of a laser beam propagating along the z-axis of a medium of constant refractive index n is

$$\nabla^2 E = \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} \ ,$$

where

$$E(x, y, z, t) = U(x, y, z)e^{i(kz-\omega t)} \ ,$$

with $U(x, y, z)$ the scalar amplitude.

a. Obtain the partial differential equation for $U(x, y, z)$.

b. A significant simplification may be made in the equation for U in part a.), amounting to neglecting a term in $\partial^2 U/\partial z^2$ but retaining a term in $\partial U/\partial z$, permitting a solution for the z-dependence of U to be given explicitly. Write down the resulting simplified equation for $U(x, y, z)$.

c. Try to give a justification for this approximation,