Classical Mechanics

September 12, 2007

Work 2 of the 3 problems. Please put each problem solution on a separate sheet of paper and put your name on each sheet.

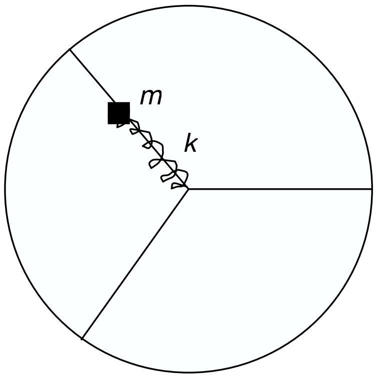
Problem 1

Find the period of a plane pendulum *without* assuming small angles. Use variables: τ for the period, g for the acceleration due to gravity, ℓ for the length of the pendulum, and θ_0 for the maximum angle. Use θ as the generalized coordinate. Find the answer to fourth order in θ_0, i.e.

$$\tau = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 \dots\right)$$

Problem 2

A flywheel of moment of inertia I rotates about its center in a horizontal plane. A mass m can slide freely along one of the spokes and is attached to the center of the wheel by a spring of natural length l_0 and force constant k, as shown in the figure:



a. Find the Lagrangean for this system.

b. Derive the equations of motion from this Lagrangean.

c. Show that the total angular momentum of the system is a constant of motion.

d. Derive an expression for the equilibrium distance, r_0, of the mass from the center of the flywheel, assuming a constant angular velocity, Ω_0.

e. Suppose that the mass has reached its equilibrium distance, r_0, for a given angular velocity, Ω_0. Now the mass is slightly displaced from its equilibrium distance, r_0. Show that the angular frequency, ω, of the small oscillations is given by:

$$\omega = \sqrt{\frac{k}{m} + \left(\frac{3mr_0^2 - I}{I + mr_0^2}\Omega_0^2\right)} \equiv \sqrt{q}$$

(Note: After the mass is released, the instantaneous angular velocity does not have to coincide with $\Omega_0 \Longrightarrow$ the replacement of $\dot{\theta}$ with Ω_0 is not justified.)

f. Explain what type of motion you find if q is negative.

Hint: $\dfrac{1}{(1+x)^2} \approx 1 - 2x$

Problem 3

A particle of mass m moves under the influence of the central potential

$$V(r) = -\frac{k}{r^4} \qquad (k > 0).$$

At time $t = 0$ the particle is at $r = r_0$ and is given a velocity of magnitude v_0 directed at an angle of 45° with respect to the radial (outward) direction. Calculate the minimum value of v_0 for which the particle will escape to infinity.
(Suggestion; Express your answer in terms of the dimensionless quantity $x = \frac{mv_0^2 r_0^4}{k}$.)